WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #5 due 11/20/2015

Problem 1. Consider the Neumann problem for the Laplacian in the 3-dimensional unit ball, that is the boundary value problem

$$\Delta u = f \text{ in } B(0,1) , \qquad \frac{\partial u}{\partial n} = g \text{ in } S^{d-1}$$

Here $B(0,1) = \{x \in \mathbb{R}^3 \ : \ |x| < 1\}$ and $S^2 = \partial B(0,1) = \{x \in \mathbb{R}^3 \ : \ |x| = 1\}$, and n is the exterior unit norm vector of $B(0,1)$ on S^2.

a.) Show that $B(0,1)$ has a C^∞ boundary.

Solution. Choose $\underline{x} = (0,0,1)$ and set $\varphi(x) = (x_1, x_2, 1-|x|)$ where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. This function is well defined for all $x \in B(0,1)$ and it is of class C^∞ for all $x \in \mathbb{R}^d \setminus \{0\}$. Also, for $|x| = 1$ that is $x \in S^2 = \partial B(0,1)$, we have $y_3 = 1 - |x| = 0$ and for $0 < |x| < 1$, that is $x \in B(0,1) \setminus \{0\}$ we have $y_3 > 0$. Let $\mathscr{U} = \mathscr{U}(\underline{x}) = \{x \in \mathbb{R}^d \ : \ |x - (0,0,1)| < 1/4\}$. Hence we have verified that $\varphi \in C^\infty(\overline{\mathscr{U}})$ and that

$$\varphi(\mathscr{U} \cap B(0,1)) \subset \{y \in \mathbb{R}^3 \ : \ y_3 > 0\} \quad \text{ and } \quad \varphi(\mathscr{U} \cap S^2) \subset \{y \in \mathbb{R}^3 \ : \ y_3 = 0\} .$$

However, one still needs to verify that φ is invertible in $\overline{\mathscr{U}}$. We have $y_3 = 1 - |x|$, which because of $y_1 = x_1$ and $y_2 - x_2$ gives $y_3 = 1 - \sqrt{1 - y_1^2 - y_2^2 - x_3^2}$. Solving for x_3 yields

$$x_3 = \sqrt{(1-y_3)^2 - y_1^2 - y_2^2} \quad \text{ and thus } \quad \varphi^{-1}(y) = \left(y_1, y_2, \sqrt{(1-y_3)^2 - y_1^2 - y_2^2} \right) .$$

Ideally, one needs to show that $\varphi^{-1} \in C^\infty(\varphi(\overline{\mathscr{U}}))$. However, the actual computation of $\varphi(\overline{\mathscr{U}})$ is tedious and can be avoided since

$$\varphi(\overline{\mathscr{U}}) \subset \{y \in \mathbb{R}^3 \ : \ y_1^2 + y_2^2 \leq 1/16 \text{ and } |y_3| \leq 1/4\} .$$

This follows from the definition of the set $\mathscr{U}$. One sees that $x \in \mathscr{U}$ implies $y_1^2 + y_2^2 < 1/16$ and $3/4 < |x < 5/4$ which results in $|y_3| < 1/4$. The function φ^{-1} is well defined and smooth on this set. A similar construction can be performed in every fixed point $\underline{x} \in S^2$

alternative solution. Certainly, there are other ways to solve this problem. Choosing $\underline{x} = (1,0,0)$, set

$$x_1 = (1-y_3) \sin y_2 \cos y_1 , \quad x_2 = (1-y_3) \sin y_2 \sin y_1 , \quad x_3 = (1-y_3) \cos y_2$$

These are pretty much the defining equations of the spherical coordinates with the two angles being denoted by y_1 and y_2 and $y_3 = 1 - r$ with $r = |x|$. One observes that we actually define the function φ^{-1}. However, the function φ can be found and is given by

$$y_1 = \cos^{-1} \frac{x_3}{|x|}, \quad y_2 = \tan^{-1} \frac{x_2}{x_1}, \quad y_3 = 1 - |x| .$$

One notes that the function φ is smooth on its domain whereas the function $\varphi^{-1} \in C^\infty(\mathbb{R}^3)$. Choosing $\mathscr{U} = \mathscr{U}(\underline{x})$ small enough guarantees $\varphi \in C^\infty(\overline{\mathscr{U}})$ and $\varphi^{-1}(\varphi(\overline{\mathscr{U}}))$. A similar construction can be given in every point of the unit sphere.

b.) For a given neighborhood of $\mathscr{U}(x)$ with $x \in S^2$ and a coordinate mapping $\varphi \in C^\infty(\mathscr{U})$ found in part a.), give an explicit transformation of this boundary value problem to the half space.

Solution. We will transform the boundary value problem to the half space using the function φ and the neighborhood $\mathscr{U}$ introduced in the first part of the problem. Given a (scalar-valued) function u supported in $\mathscr{U}$, set $v = u \circ \varphi^{-1}$, that is $v \circ \varphi = u$. By the chain rule $D_x u = D_y v D_x \varphi$ where the symbol D refers to the derivative, that is $D_x u$ is the gradient of u written as a row vector, $D_y v$ is the gradient of v (again a row vector) and $D_x \varphi$ is the Jacobian matrix of φ that is

$$D_x\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{x_1}{|x|} & \frac{x_2}{|x|} & \frac{x_3}{|x|} \end{bmatrix} .$$

Changing to column vectors we can write$\nabla_x u = [D_x\varphi]^T \nabla_y u$, or more precisely,

$$\nabla_x u(\varphi^{-1}(y)) = [D_x\varphi(\varphi^{-1}(y))]^T \nabla_y u(y)$$

and hence

$$\Delta_x u(\varphi^{-1}(y)) = \nabla_x \cdot [\nabla_x u(\varphi^{-1}(y))] = [D_x\varphi(\varphi^{-1}(y))]^T \nabla_y] \cdot [D_x\varphi(\varphi^{-1}(y))]^T \nabla_y u(y) .$$

This expression is not easily computed. However, since

$$D_x\varphi(\varphi^{-1}(y))^T = \begin{bmatrix} 1 & 0 & \frac{y_1}{1-y_3} \\ 0 & 1 & \frac{y_2}{1-y_3} \\ 0 & 0 & \frac{\sqrt{(1-y_3)^2 - y_1^2 - y_2^2}}{1-y_3} \end{bmatrix} ,$$

one observes that the principal part of $\Delta_x u(\varphi^{-1}(y))$ is of the form

$$\sum_{j,k=1}^{3} a_{jk}(y) \frac{\partial^2}{\partial y_j \partial y_k} \tag{1}$$

with a real symmetric coefficient matrix $a_{jk}(y) = [D_x\varphi(\varphi^{-1}(y)) D_x\varphi(\varphi^{-1}(y))^T]_{jk}$. Since φ is a coordinate transform, the matrix $a_j k(y)$ is uniformly positive definite on $\varphi(\mathscr{U})$. Hence, the transformed operator is again elliptic, albeit with variable coefficients.

It remains to transform the boundary condition. Recall from the lecture that

$$n(x) = \frac{[D\varphi(x)]^T e_3}{|[D\varphi(x)]^T e_3|}$$

where $e_3 = (0, 0, 1)$ is the third standard basis vector. Then for $y \in \varphi(\mathscr{U} \cap S^2)$

$$\begin{aligned} \frac{\partial u}{\partial n}(\varphi^{-1}(y)) = n(\varphi^{-1}(y)) \cdot \nabla_x u(\varphi^{-1}(y)) &= \frac{[D\varphi(\varphi^{-1}(y))]^T e_3}{|[D\varphi(\varphi^{-1}(y))]^T e_3|} \cdot [D_x\varphi(\varphi^{-1}(y))]^T \nabla_y v(y) \\ &= \frac{1}{|[D\varphi(\varphi^{-1}(y))]^T e_3|} \sum_{j=1}^{3} e_3 a_{3k}(y) \frac{\partial v}{\partial y_k}(y) . \end{aligned}$$

which shows that the Neumann boundary condition for the Laplace operator transfers - up to a factor - into the Neumann condition for the second-order operator (2). The expression

$$\sum_{j=1}^{3} e_3 a_{3k}(y) \frac{\partial v}{\partial y_k}(y)$$

is known as co-normal derivative.

Alternative solution. If one uses spherical coordinates to straighten the boundary (alternative solution to Problem 1), then one obtains the Laplacian in spherical coordinates on the half space.

Problem 2. Consider the *stationary isotropic system of elasticity* with constant coefficients in the half space, that is 3×3 system of second order

$$\mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u = f \qquad \text{in } \mathbb{R}^3_+ ,$$

where u and f are vector valued functions with 3 components each and λ and μ are real constants, called Lamé parameters.

a.) Under which conditions on μ and λ is this system elliptic ?

Solution. The (principal) symbol of the operator is the matrix

$$P_2(\xi) = \begin{bmatrix} (\mu+\lambda)\xi_1^2 + \mu|\xi|^2 & (\mu+\lambda)\xi_1\xi_2 & (\mu+\lambda)\xi_1\xi_3 \\ (\mu+\lambda)\xi_1\xi_2 & (\mu+\lambda)\xi_2^2 + \mu|\xi|^2 & (\mu+\lambda)\xi_2\xi_3 \\ (\mu+\lambda)\xi_1\xi_3 & (\mu+\lambda)\xi_2\xi_3 & (\mu+\lambda)\xi_3^2 + \mu|\xi|^2 \end{bmatrix} \xi_3$$

and the determinant is equal to $[\mu|\xi|^2]^2(2\mu+\lambda)|\xi|^2$. Hence, the system is elliptic as long as $\mu \neq 0$ and $2\mu + \lambda \neq 0$.

b.) Reduce this system to a first order system of the form $\partial v/\partial y - K(D_x)v = F$.

Solution. Start writing the symbol as a matrix polynomial in ξ_3, that is

$$\begin{aligned} P_2(\xi) = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix} \xi_3^2 + \begin{bmatrix} 0 & 0 & (\mu+\lambda)\xi_1 \\ 0 & 0 & (\mu+\lambda)\xi_2 \\ (\mu+\lambda)\xi_1 & (\mu+\lambda)\xi_2 & 0 \end{bmatrix} \xi_3 \\ + \begin{bmatrix} (\mu+\lambda)\xi_1^2 + \mu[\xi_1^2+\xi_2^2] & (\mu+\lambda)\xi_1\xi_2 & 0 \\ (\mu+\lambda)\xi_1\xi_2 & (\mu+\lambda)\xi_2^2 + \mu[\xi_1^2+\xi_2^2] & 0 \\ 0 & 0 & \mu[\xi_1^2+\xi_2^2] \end{bmatrix} . \end{aligned}$$

Hence, multiplying the principal symbol with

$$-A_2^{-1} = - \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix}^{-1}$$

results in

$$A_2^{-1} P_2(D) = \frac{\partial^2}{\partial y} + A_1(\xi_1, \xi_2) \frac{\partial}{\partial y} + A_0(\xi_1, \xi_2)$$

where $y = x_3$ with

$$A_1(\xi_1, \xi_2) = iA_2^{-1} \begin{bmatrix} 0 & 0 & (\mu+\lambda)\xi_1 \\ 0 & 0 & (\mu+\lambda)\xi_2 \\ (\mu+\lambda)\xi_1 & (\mu+\lambda)\xi_2 & 0 \end{bmatrix},$$

$$A_0(\xi_1, \xi_2) = -A_2^{-1} \begin{bmatrix} (\mu+\lambda)\xi_1^2 + \mu[\xi_1^2+\xi_2^2] & (\mu+\lambda)\xi_1\xi_2 & 0 \\ (\mu+\lambda)\xi_1\xi_2 & (\mu+\lambda)\xi_2^2 + \mu[\xi_1^2+\xi_2^2] & 0 \\ 0 & 0 & \mu[\xi_1^2+\xi_2^2] \end{bmatrix}.$$

Finally, with $v_I = \Lambda u$ and $v_{II} = \partial u/\partial y$. Here, with a slight change of notation compared to the lecture notes $\widehat{\Lambda u}(\xi) = \sqrt{1+\xi_1^2+\xi_2^2}\hat{u}(\xi)$ and hence, the 3×3 system of second order becomes a 6×6 system of first order of the form

$$\frac{\partial v}{\partial y} - K(D_{x_1}, D_{x_2})v = F$$

with

$$K(\xi_1, \xi_2) = \begin{bmatrix} 0 & \Lambda I_3 \\ A_0\Lambda^{-1} & A_1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

c.) Reduce the boundary condition $[\nabla u]^s n = g$ on $\partial\mathbb{R}^3_+ = \mathbb{R}^2$ to a boundary condition of order zero for the function v introduced in problem a.). Here $n = -e_3$ is the exterior unit normal vector (which coincides with the opposite of the last standard basis vector in $\mathbb{R}^3$) and $[\nabla u]^s$ is the 'symmetric gradient' of u, that is $[\nabla u]^s = [\nabla u + \nabla u^T]/2$. (It may be better to call this expression the symmetric Jacobian since ∇u is the Jacobian matrix of u.)

Solution. Compute

$$\nabla^s u = \frac{1}{2}\begin{bmatrix} 2\partial_1 u_1 & \partial_1 u_2 + \partial_2 u_1 & \partial_1 u_3 + \partial_3 u_1 \\ \partial_1 u_2 + \partial_2 u_1 & 2\partial_2 u_2 & \partial_3 u_2 + \partial_2 u_3 \\ \partial_1 u_3 + \partial_3 u_1 & \partial_3 u_2 + \partial_2 u_3 & 2\partial_3 u_3 \end{bmatrix}$$

and

$$[\nabla u]^s n = \frac{1}{2}\begin{bmatrix} \partial_1 u_3 + \partial_3 u_1 \\ \partial_3 u_2 + \partial_2 u_3 \\ 2\partial_3 u_3 \end{bmatrix} = g \quad \text{which gives} \quad \frac{\partial u}{\partial y} + \begin{bmatrix} \partial_1 u_3 \\ \partial_2 u_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2g_1 \\ 2g_2 \\ g_3 \end{bmatrix}$$

With the function v defined as above we have

$$v_{II} + \begin{bmatrix} 0 & 0 & \partial_1\Lambda^{-1} \\ 0 & 0 & \partial_2\Lambda^{-1} \\ 0 & 0 & 0 \end{bmatrix} v_I = \begin{bmatrix} 2g_1 \\ 2g_2 \\ g_3 \end{bmatrix}.$$

Note that this boundary condition can be written as a 3×6 matrix acting on the vector v which has 6 components.

Problem 3. *The Sobolev space $H_{(k,s)}(\mathbb{R}^d)$.* For a distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, k and $s \in \mathbb{R}$, one defines the norm

$$\|u\|^2_{(k,s)} = \int_{-\infty}^{\infty}\int_{\mathbb{R}^{d-1}} |\hat{u}(\xi,\eta)|^2(1+|\xi|^2+|\eta|^2)^k\langle\xi\rangle^{2s}d\xi d\eta$$

where $\xi = (\xi_1, ..., \xi_{d-1})$, $\langle \xi' \rangle = \sqrt{1 + |\xi'|^2}$, $\eta \in \mathbb{R}$, and $\hat{u}$ is the Fourier transform of u with respect to all d variables. Then one introduces the Sobolev space $H_{(k,s)}(\mathbb{R}^d)$ as the set

$$\{u \in \mathcal{S}'(\mathbb{R}^d) \ : \ \|u\|_{(k,s)} < \infty\} \ .$$

Show that for $k > 1/2$ there exists a linear continuous operator T from $H_{(k,s)}(\mathbb{R}^d)$ into $H^{k+s-1/2}(\mathbb{R}^{d-1})$ such that
$(Tu)(x) = u(x, 0)$ for all $u \in C_0^\infty(\mathbb{R}^d)$.

Proof. Using a density argument, it will suffice show that there exists a constant C which may depend on k and s such that

$$|u(x, 0)|_{H^{k+s-1/2}} \leq C\|u\|_{(k,s)}$$

for all $u \in C_0^\infty(\mathbb{R}^d)$.

Set $f(x) = u(x, 0)$ and let $\hat{u}(\xi, \eta)$ be the Fourier transform of u. Then

$$f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{u}(\xi, \eta) e^{ix\cdot\xi} d\xi d\eta = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} e^{ix\cdot\xi} \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\xi, \eta) d\eta \right] d\xi$$

which after Fourier transform only in the tangential variables results in

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\xi, \eta) d\eta \ .$$

Then, using the Cauchy-Schwarz inequality

$$|f(\xi)|^2 \leq \int_{\mathbb{R}} |\hat{u}(\xi, \eta)|^2 (1 + |\xi|^2 + \eta^2)^k d\eta \int_{\mathbb{R}} (1 + |\xi|^2 + \eta^2)^{-k} d\eta \tag{2}$$

where the last integral is convergent of $k > 1/2$. More precisely, with the substitution $\zeta = \eta / \langle \xi \rangle$ one obtains

$$\int_{\mathbb{R}} (1 + |\xi|^2 + \eta^2)^{-k} d\eta = \langle \xi \rangle^{-2k+1} \int_{\mathbb{R}} (1 + \zeta^2)^{-k} d\zeta = C(k) \langle \xi \rangle^{1-2k} \ .$$

After multiplying (2) by $\langle \xi \rangle^{2k+2s-1}$ and integrating over $\mathbb{R}^{d-1}$ one obtains

$$\int_{\mathbb{R}^{d-1}} \langle \xi \rangle^{2k+2s-1} |f(\xi)|^2 \leq C(k) \int_{\mathbb{R}^d} |\hat{u}(\xi, \eta)|^2 (1 + |\xi|^2 + \eta^2)^k \langle \xi \rangle^{2s} d\xi d\eta \ .$$

□